Graph Theory
Homework 4

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Proposition 0.1 (Exercise 1). *Let G be a planar graph with $n \geq 3$ vertices. The following are equivalent.*

1. *G has $m = 3n - 6$ edges.*

2. *G is maximal planar, that is, $G + xy$ is not planar for any $xy \notin E(G)$.*

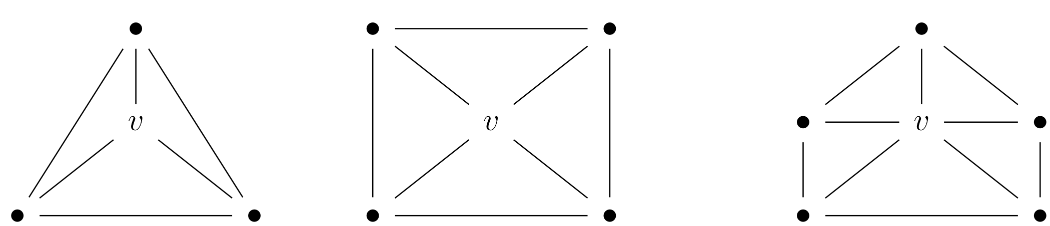
3. *G has only triangular faces, including the infinite face, that is, $\deg(F) = 3$ for all F.*

Proof. (1) $\implies$ (2). We prove the contrapositive. Suppose G is not maximal planar with n vertices and m edges, so we can add an edge e to form a planar graph $\widetilde{G}$. By Theorem 16 of Bollobas, $m + 1 \leq 3n - 6$, so $m < 3n - 6$, that is, $m \neq 3n - 6$.

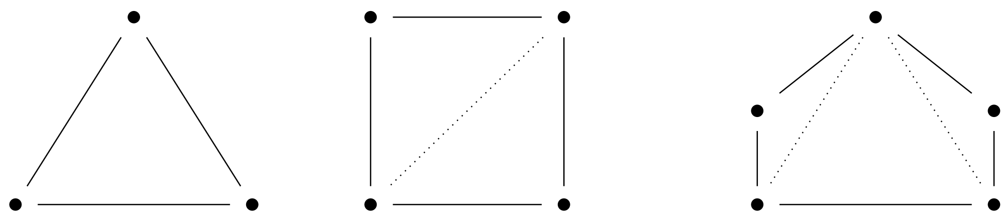
(2) $\implies$ (3). The contrapositive is "obvious." Suppose G has a face that is not triangular. Then we can add an edge to G and maintain planarity. For example,

(3) $\implies$ (2). Suppose G has only triangular faces. If xy is a possible edge addition to G, then the interior of this edge must lie in some face F of G, which is a triangle. Then the endpoints x, y must be two of the three vertices on boundary of F. But all vertices on the boundary of F already have an edge, so this is impossible. Thus G is maximal planar.

(2) $\implies$ (1). We induct on the number of vertices. The base case $n = 3$ is trivial. Let G be maximal planar with n vertices. Since G is planar, there exists a vertex v with $\deg v \leq 5$. The three possible local structures of G at v are depicted below.



Consider $G \setminus v$. Clearly, outside the neighbors of v, $G \setminus v$ is maximal planar. We can also add edges to $G \setminus v$ to make it maximal planar, depending on the degree of v. Specifically, we add exactly $\deg v - 3$ edges to make $G \setminus v$ maximal planar.



Let $\widetilde{G}$ be the new maximal planar extension of $G \setminus v$. Let $\widetilde{n}, \widetilde{m}$ be the respective vertex and edge counts for $\widetilde{G}$. Then $\widetilde{n} = n - 1$ and $\widetilde{m} = m - \deg v + (\deg v - 3) = m - 3$. By induction hypothesis, since $\widetilde{G}$ is maximal planar with fewer than n vertices, $\widetilde{m} = 3\widetilde{n} - 6$ so

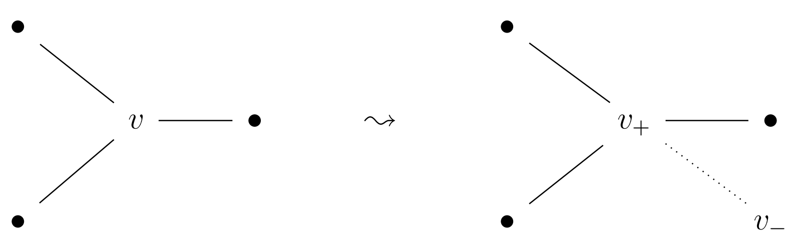
$$m - 3 = 3(n-1) - 6 \implies m = 3n - 3 - 6 + 3 = 3n - 6$$

This completes the induction. □

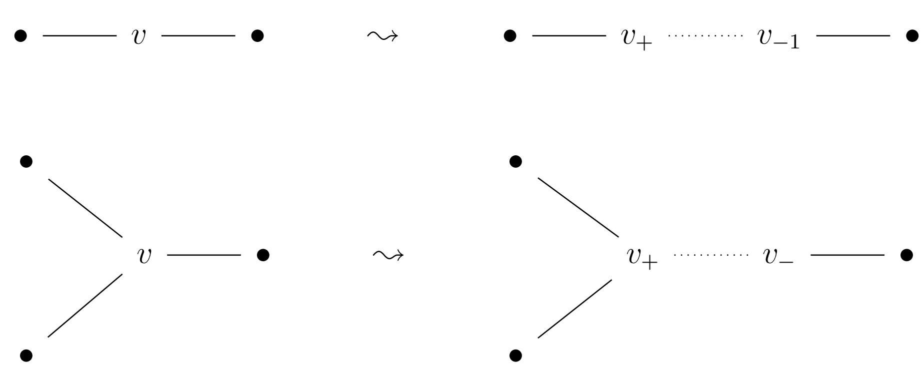
Lemma 0.2 (for Exercise 2)**.** *Let K be a graph such that $\deg v \leq 3$ for each vertex v. Then any IK contains a TK subgraph.*

Proof. We induct on the number of inflation steps. The base case of zero inflation steps is trivial. For the induction, it suffices to show that performing a single inflation on any IK containing a TK subgraph results in a graph containing a TK subgraph.

Suppose we inflate the vertex v. If v is not in TK, then this inflation does not affect TK, so IK still contains a TK subgraph, so we may assume v is in TK. Note that TK also has maximum degree 3, so at most 3 of the edges incident to v lie in TK. We will ignore edges incident to v not in TK. If our inflation just consists of adding a leaf, e.g.



then we don't change the TK subgraph. The only other possibilities for the configuration of the (up to) three TK edges incident to v after inflation are the following.



Such an inflation to TK is just the same as subdividing edges, so after inflation our IK still contains a TK subgraph. □

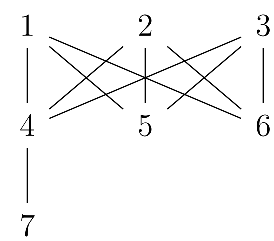
Proposition 0.3 (Exercise 2)**.** *Let K be a graph such that $\deg v \leq 3$ for each vertex v. If G is a graph such that $IK \subset G$, then $TK \subset G$.*

Proof. By the previous lemma, every IK contains a TK. □

Proposition 0.4 (Exercise 3a, part one)**.** *Let $G = (V, E)$ be the Peterson graph. If G were planar, it would violate the Edge-Region inequality, so it must be non-planar.*

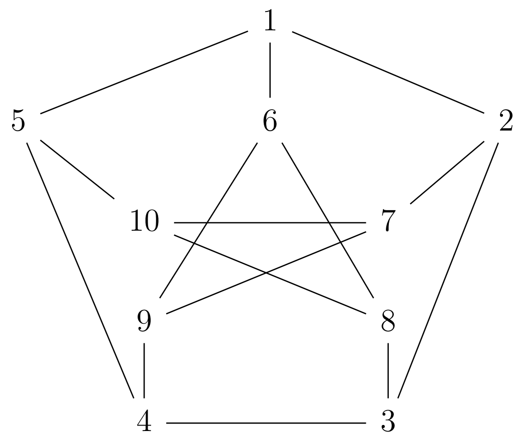
Proof. Suppose G is planar. We have $n = 10, m = 15$, so then $\ell = 2 - 10 + 15 = 7$ by Euler's formula. The girth of G is 5, so the Edge-Region Inequality says that $5(7) \leq 2(15)$ or $35 \leq 30$, which is false. Thus G is non-planar. □

Exercise 3a, part two. An example of a non-planar graph that would satisfy the Edge-Region Inequality if it were planar is the following inflation of $K_{3,3}$.



We have $n = 7, m = 10$. If the above graph were planar, it would have 5 faces by Euler's formula. The girth is 4, so $g\ell = 20 \leq 20 = 2m$. However, this graph is clearly non-planar since it is an inflation of $K_{3,3}$.

Exercise 3b We can draw the Peterson graph G as below.



We can recognize G as an IK_5 by contracting edges $(1,6),(2,7),(3,8),(4,9)$, and $(5,10)$. This helps us find the following $TK_{3,3}$ subgraph of G. We remove edges $(5,10)$ and $(2,3)$. We have replaced the extraneous vertices with dots to show the $TK_{3,3}$ structure. The underlying bipartite graph has vertex sets $\{1,8,9\}$ and $\{4,6,7\}$.